Published by Institute of Physics Publishing for SISSA
Received: *October 21, 2005*
Accepted: *November 22, 2005*
Published: *December 23, 2005*

Instantons in $\mathcal{N} = 1/2$ super Yang-Mills theory via deformed super ADHM construction

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Abstract: We study an extension of the ADHM construction to give deformed anti-self-dual (ASD) instantons in $\mathcal{N} = 1/2$ super Yang-Mills theory with U(n) gauge group. First we extend the exterior algebra on superspace to non(anti)commutative superspace and show that the $\mathcal{N} = 1/2$ super Yang-Mills theory can be reformulated in a geometrical way. By using this exterior algebra, we formulate a non(anti)commutative version of the super ADHM construction and show that the curvature two-form superfields obtained by our construction do satisfy the deformed ASD equations and thus we establish the deformed super ADHM construction. We also show that the known deformed U(2) one instanton solution is obtained by this construction.

Keywords: Solitons Monopoles and Instantons, Superspaces, Non-Commutative Geometry.

Contents

1. Introduction

In supersymmetric Yang-Mills theories, there are zero modes of adjoint fermions in the instanton background. Their existence naturally introduces the superpartner of the bosonic moduli called Grassmann collective coordinates (or fermionic moduli). The fermion zero modes together with the bosonic configurations are called super instantons. For reviews, see refs. [1, 2] for example.

It is well known that the instanton configurations of the gauge field can be obtained by the ADHM construction [3]. To give the super instanton solutions, superfield extensions of the ADHM construction were proposed in [5, 6] (see also [7]–[13]), in which the fermionic moduli belong to the same superfield containing the bosonic moduli. In the previous paper [4], we formulated the $\mathcal{N} = 1$ super ADHM construction with the use of the ‡-conjugation, and found a condition to ensure the Wess-Zumino (WZ) gauge of the gauge potential superfield (as well as the field strength superfields) obtained by this ADHM construction. The investigation of the WZ gauge is necessary if one would like to compare the results in the superfield formalism with those obtained by the component formalism. Especially it is indispensable for our present formulation of the deformed super ADHM construction in the following.

One of the motivations of our previous paper is emergence of supersymmetric gauge theory defined on a kind of deformed superspace, called non(anti)commutative superspace, in superstring theory as a low energy effective theory on D-branes with constant graviphoton field strength [14]–[17] (see [18] for earlier works on deformed superspace). In non(anti)commutative space, anticommutators of Grassmann coordinates become non-vanishing. Such a deformation of (Euclidean) four dimensional $\mathcal{N} = 1$ super Yang-Mills theory has been formulated by Seiberg [16], which is sometimes called $\mathcal{N} = 1/2$ super Yang-Mills theory. Subsequently non(anti)commutative gauge theories have been studied extensively in both perturbative and non-perturbative aspects [19]–[26].

It was argued by Imaanpur [21] that the anti-self-dual (ASD) instanton equations should be modified in the $\mathcal{N} = 1/2$ super Yang-Mills theory with self-dual (SD) non(anti)-commutativity. Solutions to those equations (deformed ASD instantons) have been studied by many authors [21]–[25] (see also [26]). In the case of U(2) gauge group, the exact one-instanton solution have been explicitly constructed in [21, 22] by perturbation with respect to the non-anticommutativity parameter. U(n) ($n \geq 2$) one-instanton solutions are obtained in [23] in a similar way. In ref. [24], the authors have studied string amplitudes in the presence of D(-1)-D3 branes with the background R-R field strength and derived constraint equations for the string modes ending on D(-1)-branes, which are nothing but the ADHM constraints for the deformed ASD instantons. On the other hand, it is far from obvious how to obtain these constraints in the purely field theoretic context and how the deformed ASD connections are given exactly in terms of the ADHM moduli parameters. Clearly we need an appropriately extended ADHM construction to answer these questions. Then it is natural to expect that useful is the superfield extension of the ADHM construction, because the field theories on non(anti)commutative superspace can be realized by deforming the multiplication of superfields.

In this paper, we extend the ADHM construction to the one that can give exact solutions to the deformed ASD equations in $\mathcal{N} = 1/2$ super Yang-Mills theory with U(n) gauge group. This is accomplished by deforming the $\mathcal{N} = 1$ super ADHM construction which we have studied in the previous paper. Our formulation provides a way to obtain other possible solutions beyond the one instanton configurations.

This paper is organized as follows. In section 2, we review $\mathcal{N} = 1/2$ super Yang-Mills theory and the deformed ASD equations. In section 3, we define a deformed exterior algebra on the non(anti)commutative superspace and show that the $\mathcal{N} = 1/2$ super Yang-Mills theory can be reproduced in a geometrical way, based on this deformed exterior algebra. In section 4, we describe a non(anti)commutative version of the $\mathcal{N} = 1$ super ADHM construction after briefly reviewing the undeformed super ADHM construction. We show that the curvature two-form superfields obtained by our construction do satisfy the deformed ASD equations. Section 5 is devoted to conclusions and discussion. In appendix A, we describe a few of our notation and conventions, although we follow our previous paper [4].[1] In appendix B, we give the "inverse" of a chiral superfield with respect to the star product, which is needed in formulating the deformed super ADHM construction. In appendix C, we give a detailed derivation of the normalized zero mode superfield of the zero-dimensional Dirac operator. In appendix D, we obtain the known U(2) one instanton solution by the deformed super ADHM construction.

2. Non(anti)commutative deformation of $\mathcal{N}=1$ super Yang-Mills

We will briefly describe the non(anti)commutative deformation of $\mathcal{N}=1$ superspace and $\mathcal{N}=1/2$ super Yang-Mills theory formulated in [16].

The non(anti)commutative deformation of $\mathcal{N}=1$ superspace is given by introducing non(anti)commutativity of the product of $\mathcal{N}=1$ superfields. This deformation is realized by the following star product:

$$f * g = f \exp(P) g\,, \qquad P = -\frac{1}{2}\overleftarrow{Q_\alpha} C^{\alpha\beta} \overrightarrow{Q_\beta}\,, \tag{2.1}$$

where f and g are $\mathcal{N}=1$ superfields and Q_α is the (chiral) supersymmetry generator. $C^{\alpha\beta}$ is the non-anticommutativity parameter and is symmetric: $C^{\alpha\beta}=C^{\beta\alpha}$. The above star product gives the following relations among the chiral coordinates $(y^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$:

$$\{\theta^\alpha, \theta^\beta\}_* = C^{\alpha\beta}\,, \qquad [y^\mu, \cdot\,]_* = 0\,, \qquad [\bar\theta^{\dot\alpha}, \cdot\,\}_* = 0\,. \tag{2.2}$$

In terms of the coordinates $(x^\mu, \theta^\alpha, \bar\theta^{\dot\alpha})$, these relations are

$$\{\theta^\alpha, \theta^\beta\}_* = C^{\alpha\beta}\,, q \quad [x^\mu, x^\nu]_* = C^{\mu\nu}\bar\theta\bar\theta\,, \qquad [x^\mu, \theta^\alpha]_* = iC^{\alpha\beta}(\sigma^\mu\bar\theta)_\beta\,, \qquad [\bar\theta^{\dot\alpha}, \cdot\,\}_* = 0\,, \tag{2.3}$$

where

$$C^{\mu\nu} \equiv C^{\alpha\beta}(\sigma^{\mu\nu})_\alpha{}^\gamma \varepsilon_{\beta\gamma}\,. \tag{2.4}$$

Since P commutes with the supercovariant spinor derivatives D_α and $\bar D_{\dot\alpha}$, the chirality notion of superfields is preserved. For example, given two chiral superfields $\Phi_i(y,\theta)$ $(i=1,2)$, the product $\Phi_1 * \Phi_2$ becomes another chiral superfields. Turning on such a deformation, the original action formulated in the $\mathcal{N}=1$ superfield formalism is deformed by the star product. Since P also commutes with Q_α, the deformed action preserves in general the chiral half of the supersymmetry transformation generated by Q_α. The deformed $\mathcal{N}=1$ super Yang-Mills theory has $\mathcal{N}=1/2$ supersymmetry, so that they are called $\mathcal{N}=1/2$ super Yang-Mills theory.

The $\ddagger$-conjugation [4] of $C^{\alpha\beta}$ is deduced from the $\ddagger$-conjugation of $\theta^\alpha\theta^\beta$ and found as

$$(C^{\alpha\beta})^\ddagger = -C_{\beta\alpha}\,, \tag{2.5}$$

[1] We should notice, however, that there is a change in the notation from our previous paper. The "anti-holomorphic" quantities with respect to the $\ddagger$-conjugation are indicated by "$\sim$" in ref. [4], while they are indicated simply by "$\ddagger$" in this paper. For example, $\tilde{\hat\Delta}_\alpha$ in [4] is denoted as $\hat\Delta^\ddagger{}_\alpha$.

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where $C_{\alpha\beta} \equiv \varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}C^{\gamma\delta}$. Let A and B are superfields. We define the ‡-conjugation of $Q_\alpha(A)$ as

$$(Q_\alpha(A))^\ddagger \equiv (-)^{|A|}(A)^\ddagger \overleftarrow{Q^\alpha}\,, \tag{2.6}$$

then we have

$$(A * B)^\ddagger = (-)^{|A||B|}(B)^\ddagger * (A)^\ddagger\,. \tag{2.7}$$

The action of $\mathcal{N} = 1/2$ supersymmetric Yang-Mills theory is given by

$$S = \frac{1}{16Ng^2}\int d^4x \left(\int d^2\theta \mathrm{tr} W^\alpha * W_\alpha + \int d^2\bar\theta \mathrm{tr}\bar W_{\dot\alpha} * \bar W^{\dot\alpha}\right), \tag{2.8}$$

where

$$W_\alpha = -\frac{1}{4}\bar D_{\dot\alpha}\bar D^{\dot\alpha}\left(e_*^{-V} * D_\alpha e_*^V\right), \qquad \bar W_{\dot\alpha} = \frac{1}{4}D^\alpha D_\alpha\left(e_*^V * \bar D_{\dot\alpha} e_*^{-V}\right), \tag{2.9}$$

and $e_*^V \equiv \sum_n \frac{1}{n!}\overbrace{V * \cdots * V}^{n}$. Here $V = V^aT^a$ with V^a the vector superfields and T^a the hermitean generators which are normalized as $\mathrm{tr}[T^aT^b] = N\delta^{ab}$. We may redefine the component fields of V in the WZ gauge such that the component gauge transformation becomes canonical (the same as the undeformed case). In [16], such a field redefinition is found to be

$$\begin{aligned} V_{\mathrm{WZ}}(y,\theta,\bar\theta) &= -\theta\sigma^\mu\bar\theta v_\mu(y) + i\theta\theta\bar\theta\bar\lambda(y) - i\bar\theta\bar\theta\theta^\alpha\left(\lambda_\alpha + \frac{1}{4}\varepsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot\gamma}\left\{\bar\lambda^{\dot\gamma}, v_\mu\right\}\right)(y) + \\ &\quad + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left(D - i\partial_\mu v^\mu\right)(y) \end{aligned} \tag{2.10}$$

and then W and $\bar W$ become

$$W_\alpha = -i\lambda_\alpha(y) + \left[\delta_\alpha^\gamma D - i(\sigma^{\mu\nu})_\alpha{}^\gamma\left(v_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar\lambda\bar\lambda\right)\right](y)\theta_\gamma + \theta\theta(\sigma^\mu\mathcal{D}_\mu\bar\lambda)_\alpha(y)\,, \tag{2.11}$$

$$\begin{aligned} \bar W_{\dot\alpha} &= i\bar\lambda_{\dot\alpha}(\bar y) + \bar\theta_{\dot\gamma}\left\{\delta_{\dot\alpha}^{\dot\gamma}D - i(\bar\sigma^{\mu\nu}\varepsilon)^{\dot\gamma}{}_{\dot\alpha}v_{\mu\nu}\right\}(\bar y) + \bar\theta\bar\theta\,\times \\ &\quad\times\left[(\mathcal{D}_\mu\lambda\sigma^\mu)_{\dot\alpha} - \frac{1}{2}C^{\mu\nu}\{v_{\mu\nu},\bar\lambda_{\dot\alpha}\} - C^{\mu\nu}\{v_\nu, \mathcal{D}_\mu\bar\lambda_{\dot\alpha} - \frac{i}{4}[v_\mu,\bar\lambda_{\dot\alpha}]\} - \frac{i}{16}|C|^2\{\bar\lambda\bar\lambda,\bar\lambda_{\dot\alpha}\}\right](\bar y), \end{aligned} \tag{2.12}$$

where $|C|^2 \equiv C^{\mu\nu}C_{\mu\nu}$. The component action is

$$S = \frac{1}{4Ng^2}\mathrm{tr}\int d^4x\left[-\frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\bar\lambda\bar\sigma^\mu\mathcal{D}_\mu\lambda + \frac{1}{2}D^2 - \frac{i}{2}C^{\mu\nu}v_{\mu\nu}\bar\lambda\bar\lambda + \frac{1}{8}|C|^2(\bar\lambda\bar\lambda)^2\right]. \tag{2.13}$$

From the component action, we can see that the equations for SD instantons are unchanged compared to the undeformed case:

$$v_{\mu\nu}^{\mathrm{ASD}} = 0\,, \qquad \bar\lambda = 0\,, \qquad \mathcal{D}_\mu\bar\sigma^\mu\lambda = 0\,, \qquad D = 0\,. \tag{2.14}$$

Therefore, the SD instanton solutions are not affected by the deformation.

On the other hand, the equations for ASD instantons should be modified. The action can be rewritten as [21]

$$S = \frac{1}{4Ng^2}\mathrm{tr}\int d^4x\left[-\frac{1}{2}\left(v_{\mu\nu}^{\mathrm{SD}} + \frac{i}{2}C_{\mu\nu}\bar\lambda\bar\lambda\right)^2 - i\bar\lambda\bar\sigma^\mu\mathcal{D}_\mu\lambda + \frac{1}{2}D^2 + \frac{1}{4}v^{\mu\nu}\tilde v_{\mu\nu}\right], \tag{2.15}$$

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where $\tilde{v}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}v_{\rho\sigma}$. From this expression, we can see that configurations which satisfies the equations of motion and is connected to the ASD instantons when turning off the deformation are the solutions to the following deformed ASD instanton equations [21]:

$$v^{\mathrm{SD}}_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar{\lambda}\bar{\lambda} = 0\,, \qquad \lambda = 0\,, \qquad \mathcal{D}_\mu\sigma^\mu\bar{\lambda} = 0\,, \qquad D = 0\,. \tag{2.16}$$

In principle, these equations can be solved perturbatively in terms of the deformation parameter C. At the zeroth order, the solutions may be given by the ordinary ADHM construction. There are right handed fermion zero modes of the Dirac operator in those ASD backgrounds. At the next order, the field strength receives the $O(C^1)$ correction from these fermion zero modes through the fermion bilinear term in the equation. This then cause the $O(C^1)$ correction to the fermion zero modes, which gives the $O(C^2)$ correction to the field strength again through the fermion bilinear. In a similar way, we can obtain the higher order corrections. In the following sections, we would like to extend the super ADHM construction to the one that can give exact solutions to the deformed ASD instanton equations (2.16) without using such a perturbative analysis with respect to C.

3. Differential forms in the deformed superspace

We will take a geometrical approach to formulate the deformed super ADHM construction by generalizing the exterior algebra: we extend the star product between superfields to the one including differential forms in superspace. This is accomplished by considering the supercharges Q_α in (2.1) as generators of supertranslation as their original definition. We will see that the deformed exterior algebra consistently leads to the $\mathcal{N} = 1/2$ super Yang-Mills theory introduced in the previous section.

3.1 Deformation of the exterior algebra

As we stated in the beginning of this section, the principle of our construction of the deformed exterior algebra is that the operators Q_α appearing in the star product are identified with the generators of supertranslation. Thus, the star product of differential forms is defined according to the representations of supersymmetry they belong to.

Since the one-form bases e^A are supertranslation invariant, we define the action of Q_α on e^A as

$$Q_\alpha(e^A) = 0\,. \tag{3.1}$$

Then for a 1-form $\omega = e^A\omega_A$, it holds that

$$Q_\alpha(\omega) = (-)^{|A|}e^A Q_\alpha(\omega_A)\,. \tag{3.2}$$

Using this action of Q_α, we define the deformed wedge product of 1-forms ω and ω' as

$$\omega \stackrel{*}{\wedge} \omega' \equiv \omega \wedge \exp\left(-\frac{1}{2}\overleftarrow{Q_\alpha}C^{\alpha\beta}\overrightarrow{Q_\beta}\right)\omega'\,, \tag{3.3}$$

where $\overleftarrow{Q}$ ($\overrightarrow{Q}$) acts on ω (ω') from the right (left) and the normal wedge product is taken for the resulting (transformed) 1-forms. Note that $\omega\overleftarrow{Q_\alpha} = (-)^{|\omega|}Q_\alpha(\omega)$. As a result, the product of supertranslation invariant 1-forms e^A is the same as the ordinary wedge product:

$$e^{A_1} \overset{*}{\wedge} e^{A_2} \overset{*}{\wedge} \cdots \overset{*}{\wedge} e^{A_p} = e^{A_1} \wedge e^{A_2} \wedge \cdots \wedge e^{A_p} \,. \tag{3.4}$$

Hereafter we will suppress the wedge symbols.

We also define the star product between a differential form ω and a superfield f as

$$\omega * f \equiv \omega \exp\left(-\frac{1}{2}\overleftarrow{Q_\alpha}C^{\alpha\beta}\overrightarrow{Q_\beta}\right) f \,, \qquad f * \omega \equiv f \exp\left(-\frac{1}{2}\overleftarrow{Q_\alpha}C^{\alpha\beta}\overrightarrow{Q_\beta}\right) \omega. \tag{3.5}$$

Then it holds that

$$[e^A, f\}_* = 0 \,, \tag{3.6}$$

where f is an arbitrary superfield.

With the use of the basis e^A, a p-form ω_p is expanded as

$$\omega_p = e^{A_1} \cdots e^{A_p} \omega_{pA_p \dots A_1} \,, \tag{3.7}$$

where the coefficients $\omega_{pA_p\dots A_1}$ are general superfields. In this basis, the product of the p- and q-form is simply given by the star product of the coefficients:

$$\omega_p * \omega_q = (-)^{(|A_1|+\cdots+|A_q|)(|B_1|+\cdots+|B_q|)} e^{A_1} \cdots e^{A_p} e^{B_1} \cdots e^{B_q} (\omega_{pA_p\dots A_1} * \omega_{qB_q\dots B_1}) \,, \tag{3.8}$$

The exterior derivative d is defined as the map from a p-form to a $p+1$-form by using the basis e^A:

$$d\omega_p = e^{A_1} \cdots e^{A_p} e^B D_B \omega_{pA_p\dots A_1} + \sum_{r-1}^{p} (-1)^{|A_{r+1}|+\cdots+|A_p|} e^{A_1} \cdots de^{A_r} \cdots e^{A_p} \omega_{pA_p\dots A_1} \tag{3.9}$$

with ω_p in eq. (3.7) and de^A is the same as the undeformed one.

Before we start to discuss the Yang-Mills theory with the use of the above differential forms, we prove the consistency of the deformed exterior algebra. In our construction the action of the exterior derivative d coincides with the undeformed case in the e^A-basis as seen from eq. (3.9). Using eqs. (3.8) and (3.7), we can prove the graded Leibniz rule:

$$d(\omega_p * \omega_q) = (-1)^q d\omega_p * \omega_q + \omega_p * d\omega_q \,. \tag{3.10}$$

It follows also that d is nilpotent: $d^2 = 0$. Finally, the associativity of the deformed exterior algebra is a direct consequence of the associativity of the star product.

Although we have used the fact that e^A (anti)commutes with superfields to derive eq. (3.8), one should notice that general differential forms do not (anti)commute with superfields. For example, we have

$$Q_\alpha(dx^\mu) = -i(\sigma^\mu d\bar\theta)_\alpha \,, \tag{3.11}$$

because dx^μ transforms as $dx^\mu \to dx^\mu - i\xi\sigma^\mu d\bar\theta$ under the supertranslation $\xi^\alpha Q_\alpha$. As a result, there are non-trivial commutators involving differential forms in $(x, \theta, \bar\theta)$-coordinates:

$$\begin{aligned} &[dx^\mu, x^\nu]_* = C^{\alpha\beta}(\sigma^\mu d\bar\theta)_\alpha(\sigma^\nu\bar\theta)_\beta \,, \qquad [dx^\mu, \theta^\alpha]_* = iC^{\alpha\beta}(\sigma^\mu d\bar\theta)_\beta \,, \qquad [dx^\mu, \bar\theta_{\dot\alpha}]_* = 0 \,, \\ &[d\theta^\alpha, f(x,\theta,\bar\theta)\}_* = [d\bar\theta_{\dot\alpha}, f(x,\theta,\bar\theta)\}_* = 0 \,, \end{aligned} \tag{3.12}$$

where $f(x, \theta, \bar\theta)$ is an arbitrary superfield.

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3.2 Reproduction of the $\mathcal{N}=1/2$ super Yang-Mills theory

In the following, we will see that the deformed wedge product defined above is consistent with the $\mathcal{N}=1/2$ super Yang-Mills theory described in the previous section, in the sense that the curvature 2-from superfield will correctly reproduce the field strength superfield W_α and $\bar{W}_{\dot\alpha}$ in (2.9) (after imposing appropriate constraints as in the undeformed case [27]) based on the deformed exterior algebra.

Given a connection 1-form superfield ϕ, the curvature superfields F_{AB} are obtained as the coefficient functions of the two-form superfield F constructed in a standard way:

$$F = d\phi + \phi * \phi \,. \tag{3.13}$$

Due to eq. (3.8), it holds that

$$\phi * \phi = (e^A\phi_A) * (e^B\phi_B) = (-)^{|A||B|}e^Ae^B(\phi_A * \phi_B) = -\frac{1}{2}e^Ae^B[\phi_B, \phi_A\}_* \,. \tag{3.14}$$

Therefore, we find the curvature superfields F_{AB} as

$$F_{AB} = D_A\phi_B - (-)^{|A||B|}D_B\phi_A - [\phi_A, \phi_B\}_* + T_{AB}{}^C\phi_C \,, \tag{3.15}$$

where $T_{AB}{}^C$ is the torsion defined by $de^C = \frac{1}{2}e^Ae^BT_{BA}{}^C$ whose non-vanishing elements are $T_{\alpha\dot\beta}{}^\mu = T_{\dot\beta\alpha}{}^\mu = 2i\sigma^\mu_{\alpha\dot\beta}$.

The proper constraints for the curvature superfields to give the $\mathcal{N}=1/2$ super Yang-Mills theory turn out to be

$$F_{\alpha\beta} = 0 \,, \qquad F_{\dot\alpha\dot\beta} = 0 \,, \qquad F_{\alpha\dot\beta} = 0 \,, \tag{3.16}$$

where the curvature superfields are given by (3.15) (see [27] for the undeformed case). We refer these constraints as the Yang-Mills constraints. Turning off the deformation, these constraints can be solved by $\phi_\alpha = -e^{-V}D_\alpha e^V$, $\phi_{\dot\alpha} = 0$, $\phi_\mu = -\frac{i}{4}\bar\sigma_\mu^{\dot\beta\beta}\bar{D}_{\dot\beta}\phi_\beta$, where V is a general superfield. This is checked with the use only of the Leibniz rule for the supercovariant derivatives, and changing the ordering of the superfields is not needed at all. Since the supercovariant derivatives D_A satisfy the Leibniz rule even in the presence of the deformation, it tells us that the spinor connection superfields of the same form as in the undeformed case are also a solution to the Yang-Mills constraints:

$$\phi_\alpha = -e_*^{-V} * D_\alpha e_*^V \,, \qquad \phi_{\dot\alpha} = 0 \,, \qquad \phi_\mu = -\frac{i}{4}\bar\sigma_\mu^{\dot\beta\beta}\bar{D}_{\dot\beta}\phi_\beta \,, \tag{3.17}$$

where V is again a general superfield.

Because of eq. (3.8), we find that the curvature superfields F_{AB} satisfy the Bianchi identities with the star product:

$$\frac{1}{2}e^Ae^Be^C\left(D_CF_{BA} - [\phi_C, F_{BA}\}_* + \frac{1}{2}T_{CB}{}^DF_{DA} + \frac{1}{2}T_{CA}{}^DF_{DB}\right) = 0 \,. \tag{3.18}$$

Determination of all the F_{AB} with the use of the Bianchi identities is completely parallel to the undeformed case, and we find

$$F_{\mu\dot\alpha} = \frac{i}{2}\mathcal{W}^\beta\sigma_{\mu\beta\dot\alpha} \,, \qquad F_{\mu\alpha} = \frac{i}{2}\sigma_{\mu\alpha\dot\beta}\bar{\mathcal{W}}^{\dot\beta} \,, \tag{3.19}$$

where $\mathcal{W}^\alpha = W^\alpha$, $\bar{\mathcal{W}}^{\dot\alpha} = e_*^{-V} * \bar{W}^{\dot\alpha} * e_*^V$ and W and $\bar{W}$ have the same forms as in the undeformed case except for every product replaced with the star product, that is, they coincide with the field strength superfields given in (2.9).

Then the invariant action with respect to super- and gauge symmetry can be constructed with the use of $\mathcal{W}$ and $\bar{\mathcal{W}}$ as in the undeformed case [27] and it is none other than the action S given in (2.8). Therefore, imposing the Yang-Mills constraints (3.16), the $\mathcal{N} = 1/2$ super Yang-Mills theory can be correctly reproduced in a geometrical way based on the deformed exterior algebra.

4. Deformed super ADHM construction

After reviewing the super ADHM construction in section 4.1, we describe its non(anti)-commutative deformation in section 4.2. The general solution obtained by the deformed construction is given in section 4.3.

4.1 Review of the $\mathcal{N} = 1$ super ADHM construction

In this subsection, we briefly review the $\mathcal{N} = 1$ super ADHM construction.

The U(n) (or SU(n)) k instanton configurations can be given by the ADHM construction [3]. Define $\Delta_\alpha(x)$ such as

$$\Delta_\alpha(x) = a_\alpha + x_{\alpha\dot\alpha} b^{\dot\alpha} \,, \tag{4.1}$$

where a_α and $b^{\dot\alpha}$ are constant $k \times (n + 2k)$ matrices and $x_{\alpha\dot\alpha} \equiv i x_\mu \sigma^\mu_{\alpha\dot\alpha}$. We assume that Δ_α has maximal rank everywhere except for a finite set of points. Its hermitean conjugate $\Delta^{\dagger\alpha} \equiv (\Delta_\alpha)^\dagger$ is given by

$$\Delta^{\dagger\alpha}(x) = a^{\dagger\alpha} + b^\dagger_{\dot\beta} x^{\dot\beta\alpha} \,. \tag{4.2}$$

Then the gauge field v_μ is given by

$$v_\mu = -2i v^\dagger \partial_\mu v \,, \tag{4.3}$$

where v is the set of the normalized zero modes of Δ_α:

$$\Delta_\alpha v = 0 \,, \qquad v^\dagger v = \mathbb{1}_n \,. \tag{4.4}$$

For later use we define f which is defined as the inverse of the quantity

$$f^{-1} \equiv \frac{1}{2} \Delta_\alpha \Delta^{\dagger\alpha} \,. \tag{4.5}$$

The super instanton condition (2.14) can be rewritten in the superfield formalism [5, 6] as

$$F_{\mu\dot\alpha} = 0 \,, \tag{4.6}$$

$$\star F_{\mu\nu} = -F_{\mu\nu} \,, \tag{4.7}$$

where F is the curvature superfield satisfying the covariant constraints (3.16) and the Bianchi identity (3.17). The super ADHM construction gives the solutions to the super ASD condition (4.6) [4]. We define a superfield extension of $\Delta_\alpha(x)$ (see also appendix A):

$$\hat{\Delta}_\alpha = \Delta_\alpha(y) + \theta_\alpha \mathcal{M}\,, \tag{4.8}$$

where $\Delta_\alpha(y)$ is the zero dimensional Dirac operator in the ordinary ADHM construction with replacing x^μ by the chiral coordinate $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and $\mathcal{M}$ is a $k \times (n+2k)$ fermionic matrix which includes the fermionic moduli. We suppose that $\hat{\Delta}_\alpha$ has a maximal rank almost everywhere as in the ordinary ADHM construction. Its $\ddagger$-conjugate $\hat{\Delta}^{\ddagger\alpha}$ is found to be

$$\hat{\Delta}^{\ddagger\alpha} = \Delta^{\dagger\alpha}(y) + \theta^\alpha \mathcal{M}^\dagger\,. \tag{4.9}$$

As $\hat{\Delta}_\alpha$ has n zero modes we collect them in a matrix superfield $\hat{v}_{[n+2k]\times[n]}$:

$$\hat{\Delta}_\alpha \hat{v} = 0\,. \tag{4.10}$$

Its $\ddagger$-conjugate $\hat{v}^\ddagger$ satisfies $\hat{v}^\ddagger\hat{\Delta}^{\ddagger\alpha} = 0$. We require that $\hat{v}$ satisfies the normalization condition:

$$\hat{v}^\ddagger\hat{v} = 1\,. \tag{4.11}$$

The connection one-form superfield ϕ is given by

$$\phi = -\hat{v}^\ddagger d\hat{v}\,, \tag{4.12}$$

where d is exterior derivative of superspace. The connection ϕ defines the curvature

$$F = d\phi + \phi\phi = \hat{v}^\ddagger d\hat{\Delta}^{\ddagger\alpha}\hat{K}_\alpha{}^\beta d\hat{\Delta}_\beta\hat{v}\,, \tag{4.13}$$

where

$$\hat{K}^{-1}{}_\alpha{}^\beta \equiv \hat{\Delta}_\alpha\hat{\Delta}^{\ddagger\beta} \tag{4.14}$$

and $\hat{K}_\alpha{}^\beta$ is defined such that $\hat{K}^{-1}{}_\alpha{}^\beta\hat{K}_\beta{}^\gamma = \hat{K}_\alpha{}^\beta\hat{K}^{-1}{}_\beta{}^\gamma = \delta_\alpha^\gamma\mathbb{1}_k$. Note that we have the following completeness condition:

$$\hat{v}\hat{v}^\ddagger = \mathbb{1}_{n+2k} - \hat{\Delta}^{\ddagger\alpha}\hat{K}_\alpha{}^\beta\hat{\Delta}_\beta\,. \tag{4.15}$$

The curvature superfield $F_{\mu\nu}$ becomes ASD if $\hat{K}$ satisfies $\hat{\Delta}_\alpha\hat{\Delta}^{\ddagger\beta} \propto \delta_\alpha^\beta$ and thus

$$\hat{K}^{-1}{}_\alpha{}^\beta = \delta_\alpha^\beta\hat{f}^{-1}\,, \tag{4.16}$$

where

$$\hat{f}^{-1} \equiv \frac{1}{2}\hat{\Delta}_\alpha\hat{\Delta}^{\ddagger\alpha} \tag{4.17}$$

is a $k\times k$ matrix superfield. There exists $\hat{f}$ because we have assumed that $\hat{\Delta}_\alpha$ has maximal rank. The above condition (4.16) leads to both the bosonic and fermionic ADHM constraints. When eq. (4.16) holds, i.e., the parameters in $\hat{\Delta}_\alpha$ are satisfying both bosonic and fermionic ADHM constraints, we obtain from eq. (4.13) the ASD curvature superfield (4.7) and another (non-trivial) one in terms of the ADHM quantities:

$$F_{\mu\nu} = 4\hat{v}^\ddagger b^\dagger \bar{\sigma}_{\mu\nu} \hat{f} b \hat{v}\,, \tag{4.18}$$

$$F_{\mu\alpha} = \frac{i}{2}\sigma_{\mu\alpha\dot{\beta}}\left\{-2\hat{v}^\ddagger(b^{\dagger\dot{\beta}}\hat{f}\mathcal{M} - \mathcal{M}^\dagger \hat{f} b^{\dot{\beta}})\hat{v} - 8\bar{\theta}_{\dot{\gamma}}\hat{v}^\ddagger(b^{\dagger\dot{\beta}}\hat{f}b^{\dot{\gamma}} + b^{\dagger\dot{\gamma}}\hat{f}b^{\dot{\beta}})\hat{v}\right\}. \tag{4.19}$$

We can check that the other curvature superfields vanish.

To ensure that the superfields obtained by the super ADHM construction are correctly in the WZ gauge, we impose the following conditions on $\hat{v}$:

$$\bar{D}_{\dot{\alpha}}\hat{v} = 0\,, \qquad \hat{v}^\ddagger \frac{\partial}{\partial\theta^\alpha}\hat{v} = 0\,. \tag{4.20}$$

Imposing these conditions, we can determine the zero mode $\hat{v}$ of $\hat{\Delta}_\alpha$ as

$$\hat{v} = v + \theta^\gamma\Big(\Delta^\dagger_\gamma f\mathcal{M}v\Big) + \theta\theta\Big(\frac{1}{2}\mathcal{M}^\dagger f\mathcal{M}v\Big)\,, \tag{4.21}$$

and find that the connection superfield ϕ_μ constructed as in (4.12) correctly gives the super instanton configuration in the WZ gauge:

$$\phi_\mu = -\frac{i}{2}\left[-2iv^\dagger\partial_\mu v + i\theta^\gamma\sigma_{\mu\gamma\dot{\beta}}\left\{2iv^\dagger(b^{\dagger\dot{\beta}}f\mathcal{M} - \mathcal{M}^\dagger f b^{\dot{\beta}})v\right\}\right]\,, \tag{4.22}$$

where the lowest component is the instanton gauge field and the θ-component is the fermion zero mode.

4.2 Deformation of the super ADHM construction

In this subsection, extending the super ADHM construction in section 4.1, we will present a formulation that provides a way to construct deformed ASD instantons in the non(anti)-commutative $\mathcal{N} = 1/2$ super Yang-Mills theory, i.e. the exact solutions to the deformed equations (2.16).

The deformed super ASD condition turns out to be of the same form as the super ASD condition (4.6) but the product replaced with the star product (2.1):

$$F_{\mu\dot{\alpha}} = 0\,, \tag{4.23}$$

$$\star F_{\mu\nu} = -F_{\mu\nu}\,, \tag{4.24}$$

where the curvature superfields F_{AB} are given by eq. (3.15). Note that eq. (4.24) follows from eq. (4.23) as long as the two-form F satisfies the Bianchi identities and the Yang-Mills constraints, because these imply that $F_{\mu\nu} = -\frac{1}{4}(\bar{\mathcal{D}}\bar{\sigma}_{\mu\nu}\bar{\mathcal{W}} - \mathcal{D}\sigma_{\mu\nu}\mathcal{W})$ and also that $\mathcal{W}$ is proportional to $F_{\mu\dot{\alpha}}$ (see eq. (3.19)).

Since the equivalence of the condition (4.23) and the deformed equations (2.16) is not apparent, we will show it below. We assume that the two-form F satisfies the Bianchi identities and the Yang-Mills constraints. Then $F_{\mu\dot{\alpha}}$ is proportional to W_α given in (2.9) (or (2.11) in the WZ gauge) as discussed before (see eq. (3.19)). If the deformed equations (2.16) are satisfied, we immediately find from eq. (2.11) that $W_\alpha = 0$ holds, and this implies that eq. (4.23) also holds because of eq. (3.19). The converse can be shown as follows. First let us assume $\phi_{\dot\alpha} = 0$. In this case, a solution ϕ_A to the Yang-Mills constraints can always be written of the form given in (3.17) with V a general superfield. Then $\mathcal{W}$ is determined by eq. (3.19) and will coincide with the field strength superfield W in (2.9). Taking the WZ gauge for V by performing gauge transformations, $\mathcal{W}(= W)$ becomes to have the form given in (2.11) provided that the component fields of V are parameterized as in (2.10). Now it is obvious that in order for $F_{\mu\dot\alpha}$ ($\propto \mathcal{W}$) to vanish, the deformed ASD equations should hold. Since all other solutions ϕ_A (with $\phi_{\dot\alpha} \neq 0$) to the Yang-Mills constraints are obtained from (3.17) by gauge transformation

$$\phi \to X^{-1} * \phi * X - X^{-1} * dX\,, \qquad F \to X^{-1} * F * X \tag{4.25}$$

with X an arbitrary invertible ($\mathrm{U}(n)$ valued) superfield, the equivalence also holds even in the absence of the assumption $\phi_{\dot\alpha} = 0$.

Let us remark on an implication of the second condition (4.24). eq. (4.24) requires especially that the lowest component of $F_{\mu\nu}$ contains only the ASD part, but this does not mean $v^{\mathrm{SD}}_{\mu\nu} = 0$, since the lowest component is not simply $v_{\mu\nu}$ but $v_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar\lambda\bar\lambda$ in the deformed theory: Using eq. (3.17), we find ϕ_μ in the WZ gauge as

$$\phi_\mu = -\frac{i}{2}\left[v_\mu + i\theta\sigma_\mu\bar\lambda - \bar\theta\bar\sigma_\mu W\right](y)\,. \tag{4.26}$$

Here we have used V_{WZ} given in (2.10). Then the curvature $F_{\mu\nu}$ is found to be

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu\phi_\nu - \partial_\nu\phi_\mu - [\phi_\mu, \phi_\nu]_* \\ &= -\frac{i}{2}\left(v_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar\lambda\bar\lambda - i\theta\sigma_\mu\mathcal{D}_\nu\bar\lambda + i\theta\sigma_\nu\mathcal{D}_\mu\bar\lambda - i\theta\theta\bar\lambda\bar\sigma_{\mu\nu}\bar\lambda\right) + \\ &\quad + (\text{terms containing } \bar\theta \text{ and } W). \end{aligned} \tag{4.27}$$

Therefore, we can see that in order for $F_{\mu\nu}$ to satisfy the ASD condition, at least $v^{\mathrm{SD}}_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\bar\lambda\bar\lambda = 0$ is required. In fact, if we use the deformed ASD equations (2.16) (therefore $W^\alpha = 0$), $F_{\mu\nu}$ reduces to

$$F_{\mu\nu} = -\frac{i}{2}\left(v^{\mathrm{ASD}}_{\mu\nu} - i\theta^\alpha\sigma^\rho\bar\sigma_{\mu\nu}\mathcal{D}_\rho\bar\lambda - i\theta\theta\bar\lambda\bar\sigma_{\mu\nu}\bar\lambda\right), \tag{4.28}$$

and now it contains only ASD components.

We have seen that the deformed ASD equations are equivalent to the super ASD condition with the star product, (4.23). One would expect that its solutions can be constructed by the super ADHM construction, replacing each product with the star product (2.1). In the rest of this subsection, we will see that such a deformed ADHM construction actually gives solutions to the deformed super ASD condition, that is, the deformed super ASD instantons.

For the deformed super ASD instantons, ϕ_μ in the WZ gauge becomes

$$\phi_\mu = -\frac{i}{2}\left[v_\mu + i\theta\sigma_\mu\bar\lambda\right](y)\,, \tag{4.29}$$

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since $W_\alpha = 0$ holds. This again leads us to adopt $\hat{\Delta}_\alpha$ in our super ADHM construction with the same form as in the undeformed case:

$$\hat{\Delta}_\alpha = \Delta_\alpha(y) + \theta_\alpha \mathcal{M} \,. \tag{4.30}$$

Then, according to the ‡-conjugation rules, we have

$$\hat{\Delta}^{\ddagger\alpha} = \Delta^{\ddagger\alpha}(y) + \theta^\alpha \mathcal{M}^\ddagger \,. \tag{4.31}$$

Here we will not rewrite ‡ in the r.h.s. with †, because in the presence of the deformation, Δ_α (and possibly $\mathcal{M}$) may contain products of Grassmann variables and for such quantities we should use ‡ instead of † in general.

We collect the n zero modes of $\hat{\Delta}$ into a matrix form $\hat{u}_{[n+2k]\times[n]}$:

$$\hat{\Delta}_\alpha * \hat{u} = 0 \,. \tag{4.32}$$

We require it to be normalized as

$$\hat{u}^\ddagger * \hat{u} = \mathbb{1}_n \,. \tag{4.33}$$

Define $k\times k$ matrices $\hat{K}_{*\alpha}{}^\beta$ $(\alpha,\beta=1,2)$ as the "inverse" matrices of

$$\hat{K}^{-1}_{*}{}_\alpha{}^\beta \equiv \hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta} \tag{4.34}$$

such that $\hat{K}^{-1}_{*}{}_\alpha{}^\beta * \hat{K}_{*\beta}{}^\gamma = \hat{K}_{*\alpha}{}^\beta * \hat{K}^{-1}_{*}{}_\beta{}^\gamma = \delta_\alpha^\gamma \mathbb{1}_k$ (see appendix B). Then we have the relation

$$\hat{u} * \hat{u}^\ddagger = \mathbb{1}_{n+2k} - \hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * \hat{\Delta}_\beta \,. \tag{4.35}$$

With the use of $\hat{u}$, the connection ϕ is given by

$$\phi = -\hat{u}^\ddagger * d\hat{u} \,. \tag{4.36}$$

The curvature two-form is given by using the connection one-form ϕ, and now it is written as

$$F = d\phi + \phi * \phi = \hat{u}^\ddagger * d\hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * d\hat{\Delta}_\beta * \hat{u} \,. \tag{4.37}$$

Here we have used (4.33), (4.35) and (4.32). The above equation reads

$$F_{AB} = -\hat{u}^\ddagger * D_{[A}\hat{\Delta}^{\ddagger\alpha} * \hat{K}_{*\alpha}{}^\beta * D_{B\}}\hat{\Delta}_\beta * \hat{u} \,. \tag{4.38}$$

From this equation, we find

$$F_{\mu\nu} = \hat{u}^\ddagger * b^\dagger_{\dot\alpha} \bar\sigma_{[\mu}{}^{\dot\alpha\alpha} \hat{K}_{*\alpha}{}^\beta \sigma_{\nu]\beta\dot\beta} b^{\dot\beta} * \hat{u} \,. \tag{4.39}$$

Thus the ASD condition (4.24) is satisfied if $\hat{K}_*$ commutes with the Pauli matrices:

$$\hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta} = \hat{K}^{-1}_{*}{}_\alpha{}^\beta \propto \delta_\alpha^\beta \,. \tag{4.40}$$

We immediately find from the expression (4.38) that $F_{\dot\alpha\dot\beta} = F_{\alpha\dot\beta} = 0$ and $F_{\mu\dot\alpha} = 0$, because $\hat{\Delta}_\alpha$ is a chiral superfield. We can also check that $F_{\alpha\beta} = 0$ with the use of the constraint (4.40), the relations

$$D_\beta\hat{\Delta}_\alpha = \varepsilon_{\alpha\beta}(\mathcal{M} + 4\bar\theta_{\dot\beta} b^{\dot\beta}), \quad D_\beta\hat{\Delta}^{\ddagger\alpha} = \delta_\beta^\alpha(\mathcal{M}^\ddagger + 4b^\dagger_{\dot\beta}\bar\theta^{\dot\beta}) \,, \tag{4.41}$$

and the fact that $F_{\alpha\beta}$ is symmetric with respect to α and β. Therefore, we have shown that the above described super ADHM construction gives curvature superfields that satisfy the Yang-Mills constraints (3.16) and the ASD conditions (4.23)–(4.24) if the condition (4.40) is imposed.

The requirement (4.40) gives the deformed bosonic and fermionic ADHM constraints as we will see below. Because we can write

$$\hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta} = \hat{\Delta}_\alpha \hat{\Delta}^{\ddagger\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma} C^{\gamma\beta} \mathcal{M}\mathcal{M}^\ddagger , \tag{4.42}$$

we find that the requirement leads to the deformed bosonic ADHM constraint

$$\Delta_\alpha \Delta^{\ddagger\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma} C^{\gamma\beta} \mathcal{M}\mathcal{M}^\ddagger \propto \delta_\alpha^\beta , \tag{4.43}$$

and the fermionic ADHM constraint

$$\Delta_\alpha \mathcal{M}^\ddagger + \mathcal{M}\Delta^\ddagger{}_\alpha = 0 . \tag{4.44}$$

The above equations can also be written as follows (see appendix A):

$$a'^{\ddagger}_\mu = a'_\mu , \qquad \sigma^i{}_\beta{}^\alpha \left(a_\alpha a^{\ddagger\beta} - \frac{1}{2}\varepsilon_{\alpha\gamma} C^{\gamma\beta} \mathcal{M}\mathcal{M}^\ddagger \right) = 0 , \tag{4.45}$$

$$\mathcal{M}'^{\ddagger\dot\alpha} = \mathcal{M}'^{\dot\alpha} , \qquad a_\alpha \mathcal{M}^\ddagger = -\mathcal{M} a^\ddagger_\alpha . \tag{4.46}$$

These constraints agree with those in [24] obtained by considering string amplitudes. We can rewrite the deformed ADHM constraints in another form as follows. Let us denote

$$\begin{pmatrix} \hat{\Delta}_1 \\ \hat{\Delta}_2 \end{pmatrix} = \begin{pmatrix} \hat{J}^\ddagger_{[k]\times[n]} & \bar{z}_2 \mathbb{1}_k + \hat{B}_2{}^\ddagger_{[k]\times[k]} & \bar{z}_1 \mathbb{1}_k + \hat{B}_1{}^\ddagger_{[k]\times[k]} \\ \hat{I}_{[k]\times[n]} & -z_1 \mathbb{1}_k - \hat{B}_{1[k]\times[k]} & z_2 \mathbb{1}_k + \hat{B}_{2[k]\times[k]} \end{pmatrix} , \tag{4.47}$$

where $z_1 \equiv y_{2\dot1}$, $z_2 \equiv y_{2\dot2}$ and

$$\begin{aligned} &\hat{I} \equiv I + \theta^1 \mu , && \hat{J} \equiv J + \theta^1 \mu^\ddagger , && \hat{B}_1 \equiv B_1 - \theta^1 \mathcal{M}'_{\dot1} , && \hat{B}_2 \equiv B_2 + \theta^1 \mathcal{M}'_{\dot2} , \\ &I \equiv \omega_2 , && J^\ddagger \equiv \omega_1 , && B_1 \equiv a'_{2\dot1} , && B_2 \equiv a'_{2\dot2} . \end{aligned} \tag{4.48}$$

Here we have already used $a'^{\ddagger}_\mu = a'_\mu$ and $\mathcal{M}'^{\ddagger\dot\alpha} = \mathcal{M}'^{\dot\alpha}$. Then the constraint (4.40) reads

$$\hat{I} * \hat{I}^\ddagger - \hat{J}^\ddagger * \hat{J} + [\hat{B}_1, \hat{B}_1{}^\ddagger]_* + [\hat{B}_2, \hat{B}_2{}^\ddagger]_* = 0 , \tag{4.49}$$

$$\hat{I} * \hat{J} + [\hat{B}_2, \hat{B}_1]_* = 0 . \tag{4.50}$$

In the component language, we find that the bosonic ADHM constraints are

$$II^\ddagger - J^\ddagger J + [B_1, B_1{}^\ddagger] + [B_2, B_2{}^\ddagger] - C^{12}\mathcal{M}\mathcal{M}^\ddagger = 0 , \tag{4.51}$$

$$IJ + [B_2, B_1] - \frac{1}{2} C^{11} \mathcal{M}\mathcal{M}^\ddagger = 0 , \tag{4.52}$$

and the fermionic ADHM constraints are

$$J^\ddagger \mu^\ddagger - \mu I^\ddagger - [B_1{}^\ddagger, \mathcal{M}'_{\dot1}] + [B_2{}^\ddagger, \mathcal{M}'_{\dot2}] = 0 , \tag{4.53}$$

$$\mu J + I\mu^\ddagger - [B_1, \mathcal{M}'_{\dot2}] - [B_2, \mathcal{M}'_{\dot1}] = 0 . \tag{4.54}$$

After imposing the ADHM constraints, we can write $\hat{K}_*^{-1}{}_\alpha{}^\beta = \hat{\Delta}_\alpha * \hat{\Delta}^{\ddagger\beta}$ as

$$\hat{K}_*^{-1}{}_\alpha{}^\beta = \delta_\alpha^\beta \hat{f}^{-1}\,, \tag{4.55}$$

where a $k \times k$ matrix valued superfield $\hat{f}^{-1}$ is defined by

$$\hat{f}^{-1} \equiv \frac{1}{2}\hat{\Delta}_\alpha \hat{\Delta}^{\ddagger\alpha}\,, \tag{4.56}$$

which is the same form as in the undeformed case since eq. (4.42) holds. As a result, the curvature two-form is written as

$$F = \hat{u}^\ddagger * d\hat{\Delta}^{\ddagger\alpha} * \hat{f}_* * d\hat{\Delta}_\alpha * \hat{u}\,, \tag{4.57}$$

where $\hat{f}_*$ is defined such that $\hat{f}_* * \hat{f}^{-1} = \hat{f}^{-1} * \hat{f}_* = \mathbb{1}_k$ (see appendix B). Substituting the form of $\hat{\Delta}$, we find the following equations from the above expression:

$$F_{\alpha\beta} = F_{\dot\alpha\dot\beta} = F_{\alpha\dot\beta} = 0\,, \tag{4.58}$$

$$F_{\mu\nu} = 4\hat{u}^\ddagger * b^\dagger \bar\sigma_{\mu\nu} \hat{f}_* b * \hat{u}\,, \tag{4.59}$$

$$F_{\mu\alpha} = \frac{i}{2}\sigma_{\mu\alpha\dot\beta}\left\{-2\hat{u}^\ddagger * (b^{\dagger\dot\beta}\hat{f}_*\mathcal{M} - \mathcal{M}^\ddagger \hat{f}_* b^{\dot\beta}) * \hat{u} - 8\bar\theta_{\dot\gamma}\hat{u}^\ddagger * (b^{\dagger\dot\beta}\hat{f}_* b^{\dot\gamma} + b^{\dagger\dot\gamma}\hat{f}_* b^{\dot\beta}) * \hat{u}\right\}, \tag{4.60}$$

$$F_{\mu\dot\alpha} = 0\,. \tag{4.61}$$

As mentioned before, the curvature two-form F satisfies the Yang-Mills constraints (3.16) and the deformed super ASD condition (4.23) (4.24), thus it gives the deformed super ASD instantons.

Owing to the deformed ADHM construction, we are able to discuss the dimension of the moduli space of the deformed instanton solution. The curvature F is invariant under the following $\mathrm{GL}(k) \times \mathrm{U}(n+2k)$ global symmetry transformation

$$\hat{\Delta}_\alpha \to G\hat{\Delta}_\alpha\Lambda\,, \qquad \hat{f} \to G\hat{f}G^\dagger\,, \qquad \hat{v} \to \Lambda^{-1}\hat{v}\,, \tag{4.62}$$

where $G \in \mathrm{GL}(k)$ and $\Lambda \in \mathrm{U}(n+2k)$. Note that we cannot take G and Λ as superfields, since the form of $\hat{\Delta}_\alpha$ is significant for the two-form F (4.38) to satisfy the Yang-Mills constraints as well as the deformed super ASD condition. After fixing b in the canonical form (see (A.10)), the global symmetry breaks down to $\mathrm{U}(n) \times \mathrm{U}(k)$ as in the purely bosonic ADHM construction, and the $\mathrm{U}(n)$ transformation is considered as a part of the gauge transformation. Therefore, the number of the bosonic moduli contained in $\hat{\Delta}_\alpha$ is $4nk$ after imposing the bosonic ADHM constraints (4.45) and modding out by the $\mathrm{U}(k)$ symmetry, as in the undeformed case. There is no additional symmetry and the number of fermionic parameters is reduced simply by the fermionic ADHM constraints (4.46) and we have $2kn$ fermionic moduli as in the undeformed case.

4.3 The general solution in the Wess-Zumino gauge

In this subsection, we give an expression in terms of the ADHM data Δ_α and $\mathcal{M}$, of the general solution in the WZ gauge obtained by our construction.

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In the WZ gauge, $F_{\mu\nu}$ is a chiral superfield because ϕ_μ is so. Since we are interested in the field strength in the WZ gauge, we find from the expression (4.59) that it is sufficient to restrict the zero mode $\hat{u}$ to a chiral superfield. Hereafter we restrict the zero mode $\hat{u}$ to a chiral superfield, which we write as

$$\hat{u} = u^{(0)} + \theta^\gamma u^{(1)}_\gamma + \theta\theta u^{(2)} . \tag{4.63}$$

When $\hat{u}$ and $\hat{u}^\ddagger$ are chiral superfields, we find from (4.36) that $\phi^{\dot{\alpha}} = 0$, and that ϕ_μ is a chiral superfield ($\bar{D}_{\dot{\alpha}}\phi_\mu = 0$) because ϕ_μ is given by

$$\phi_\mu = -\hat{u}^\ddagger * \frac{\partial}{\partial y^\mu}\hat{u}(y,\theta) . \tag{4.64}$$

These are consistent with the connection superfields for the super instantons. The rest of the connection ϕ_α gives a non-trivial necessary condition. In the ADHM construction, ϕ_α can be written in the chiral basis as

$$\phi_\alpha = -\hat{u}^\ddagger * D_\alpha \hat{u} = -\hat{u}^\ddagger * \frac{\partial}{\partial\theta^\alpha}\hat{u}(y,\theta) - 2i(\sigma^\mu\bar{\theta})_\alpha \hat{u}^\ddagger * \frac{\partial}{\partial y^\mu}\hat{u}(y,\theta) . \tag{4.65}$$

We should notice that only the first term is $\bar{\theta}$-independent. In the WZ gauge, ϕ_α is given by

$$\phi_\alpha = -e_*^{-V_{\rm WZ}} * D_\alpha e_*^{V_{\rm WZ}} = -D_\alpha V_{\rm WZ} + \frac{1}{2}[V_{\rm WZ}, D_\alpha V_{\rm WZ}]_* . \tag{4.66}$$

Because $V_{\rm WZ}$ contains at least one $\bar{\theta}$ in each term, ϕ_α should not have any $\bar{\theta}$-independent terms. As a result, it should hold that

$$\hat{u}^\ddagger * \frac{\partial}{\partial\theta^\alpha}\hat{u}(y,\theta) = 0 , \tag{4.67}$$

which is a necessary condition for $\hat{u}$ to be in the WZ gauge. We can use this condition to determine $\hat{u}$ in the WZ gauge.

For convenience, let us define a $k \times k$ matrix

$$K^{-1}{}_\alpha{}^\beta \equiv \Delta_\alpha \Delta^{\ddagger\beta} \tag{4.68}$$

and its inverse K such that $K_\alpha{}^\gamma K^{-1}{}_\gamma{}^\beta = K^{-1}{}_\alpha{}^\gamma K_\gamma{}^\beta = \delta_\alpha^\beta \mathbb{1}_k$. After imposing the ADHM constraint, we have

$$K^{-1}{}_\alpha{}^\beta = \delta_\alpha^\beta f^{-1} + \frac{1}{2}\varepsilon_{\alpha\gamma}C^{\gamma\beta}\mathcal{M}\mathcal{M}^\ddagger , \qquad f^{-1} \equiv \frac{1}{2}\Delta_\gamma\Delta^{\ddagger\gamma} , \tag{4.69}$$

where f^{-1} is defined as the lowest component of $\hat{f}^{-1}$ in (4.56). Defining a matrix

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_1{}^1 & \mathcal{C}_1{}^2 \\ \mathcal{C}_2{}^1 & \mathcal{C}_2{}^2 \end{pmatrix} , \quad \mathcal{C}_\alpha{}^\beta \equiv \frac{1}{2}\varepsilon_{\alpha\gamma}C^{\gamma\beta} , \tag{4.70}$$

we can write $K^{-1}{}_\alpha{}^\beta = (\mathbb{1}_2 \otimes \mathbb{1}_n + \mathcal{C} \otimes \mathcal{M}\mathcal{M}^\ddagger f)_\alpha{}^\beta f^{-1}$ where f is the inverse of f^{-1} such that $ff^{-1} = f^{-1}f = \mathbb{1}_k$. Then we find an expression of the matrix K:

$$K_\alpha{}^\beta = f\Big(\mathbb{1}_n + \det\mathcal{C}(\mathcal{M}\mathcal{M}^\ddagger f)^2\Big)^{-1}\Big\{\mathbb{1}_2 \otimes \mathbb{1}_n - \mathcal{C} \otimes \mathcal{M}\mathcal{M}^\ddagger f\Big\}_\alpha{}^\beta . \tag{4.71}$$

Here we have used a relation $\mathcal{C}_\alpha{}^\gamma\mathcal{C}_\gamma{}^\beta = -\delta_\alpha^\beta \det\mathcal{C}$.[2]

[2]From this expression we find the following useful relations:

$$K_\alpha{}^\beta\mathcal{C}_\beta{}^\gamma = \mathcal{C}_\alpha{}^\beta K_\beta{}^\gamma , \quad \mathcal{C}_\alpha{}^\gamma K_\gamma{}^\delta \mathcal{C}_\delta{}^\beta = -\det\mathcal{C} K_\alpha{}^\beta .$$

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With a given v that satisfies $\Delta_\alpha v = 0$ and $v^\ddagger v = \mathbb{1}_n$, we can determine the zero mode superfield $\hat{u}$:

$$\hat{u} = \Big(\mathbb{1}_{n+2k} + \theta^\gamma \Delta^\ddagger{}_\gamma f \mathcal{M} + \theta\theta \frac{1}{2} \mathcal{M}^\ddagger f \mathcal{M}\Big) u^{(0)} , \tag{4.72}$$

where

$$u^{(0)} = \Big\{ v - \frac{1}{2} (\Delta^\ddagger K \varepsilon C \Delta) \mathcal{M}^\ddagger Z^{-1} f \mathcal{M} v \Big\} N^{-1/2} U , \tag{4.73}$$

U is a unitary matrix such that $U^\dagger = U^{-1}$, K is given by (4.71) and Z, N are

$$Z \equiv \mathbb{1}_k + \frac{1}{2} f \mathcal{M} (\Delta^\ddagger K \varepsilon C \Delta) \mathcal{M}^\ddagger , \tag{4.74}$$

$$N \equiv \mathbb{1}_n + \frac{1}{4} \det C v^\ddagger \mathcal{M}^\ddagger f Z^{-1\ddagger} \mathcal{M} v v^\ddagger \mathcal{M}^\ddagger Z^{-1} f \mathcal{M} v . \tag{4.75}$$

Readers can check that the above $\hat{u}$ is indeed a normalized zero mode of $\hat{\Delta}_\alpha$, satisfying the WZ gauge condition (4.67) (for the detailed derivation, see appendix C). We have used the normalized zero mode v of Δ_α and it will be found as follows. If we solve the deformed ADHM constraints, the zero dimensional Dirac operator Δ_α (the fermionic moduli $\mathcal{M}$) can be separated into the C-independent part $\Delta_{0\alpha}$ ($\mathcal{M}_0$) and the residual C-dependent part $\delta\Delta_\alpha$ ($\delta\mathcal{M}$):

$$\Delta_\alpha = \Delta_{0\alpha} + \delta\Delta_\alpha , \qquad \mathcal{M} = \mathcal{M}_0 + \delta\mathcal{M} . \tag{4.76}$$

Then the zero mode v of Δ_α is given by

$$\begin{aligned} v = {} & (\mathbb{1}_{n+2k} + \Delta_0^{\dagger\alpha} f_0 \delta\Delta_\alpha)^{-1} v_0 \times \\ & \times \Big\{ v_0^\dagger (\mathbb{1}_{n+2k} + \delta\Delta^{\ddagger\gamma} f_0 \Delta_{0\gamma})^{-1} (\mathbb{1}_{n+2k} + \Delta_0^{\dagger\gamma} f_0 \delta\Delta_\gamma)^{-1} v_0 \Big\}^{-1/2} , \end{aligned} \tag{4.77}$$

where v_0 satisfies $\Delta_{0\alpha} v_0 = 0$ as well as the completeness relation $v_0 v_0^\dagger = \mathbb{1}_{n+2k} - \Delta_0^{\dagger\alpha} f_0 \Delta_{0\alpha}$. Here $f_0^{-1} \equiv \frac{1}{2} \Delta_{0\gamma} \Delta_0^{\dagger\gamma}$ and f_0 is its inverse matrix. We can check that the above $\hat{u}$ satisfies simultaneously the zero mode equation (4.32), the normalization condition (4.33) and the WZ gauge condition (4.67).

The connection superfield ϕ_μ is constructed with the use of $\hat{u}$ in (4.72):

$$\begin{aligned} \phi_\mu = {} & -u^{(0)\ddagger} \partial_\mu u^{(0)} + \\ & + \frac{1}{2} u^{(0)\ddagger} \mathcal{M}^\ddagger f (\varepsilon C)_\beta{}^\alpha \Delta_\alpha \partial_\mu \Delta^{\ddagger\beta} f \mathcal{M} u^{(0)} - \frac{1}{4} \det C u^{(0)\ddagger} \mathcal{M}^\ddagger f \mathcal{M} \mathcal{M}^\ddagger \partial_\mu \Big(f \mathcal{M} u^{(0)} \Big) - \\ & - \theta^\alpha u^{(0)\ddagger} (\partial_\mu \Delta^\ddagger{}_\alpha f \mathcal{M} + \mathcal{M}^\ddagger f \partial_\mu \Delta_\alpha) u^{(0)} . \end{aligned} \tag{4.78}$$

Here we have used the following relations:

$$\Delta_\alpha u^{(0)} + \frac{1}{2} (\varepsilon C)_\alpha{}^\beta \Delta_\beta \mathcal{M}^\ddagger f \mathcal{M} u^{(0)} = 0 , \tag{4.79}$$

$$\partial_\mu \Delta_\alpha \mathcal{M}^\ddagger + \mathcal{M} \partial_\mu \Delta^\ddagger{}_\alpha = 0 , \tag{4.80}$$

$$\partial_\mu f = f \Delta^\gamma \partial_\mu \Delta^\ddagger{}_\gamma f . \tag{4.81}$$

The first equation can be shown with the use of eq.$\tilde{(}$4.73) and

$$Z^{-1} = \mathbb{1}_k - \frac{1}{2} f \mathcal{M} (\Delta^{\ddagger} K \varepsilon C \Delta) \mathcal{M}^{\ddagger} Z^{-1} \,.$$

The last equation follows from the bosonic ADHM constraints.

Now, because of the WZ gauge, we are able to compare our connection one-form to the solutions obtained in the component formalism. In appendix D, we show that the known U(2) one instanton solution is obtained by our construction.

5. Conclusions and discussion

In this paper, we have extended the super ADHM construction to give solutions to the deformed ASD instanton equations in $\mathcal{N} = 1/2$ super Yang-Mills theory with U(n) gauge group.

First we have extended the exterior algebra on superspace to non(anti)commutative superspace, and shown that it is a consistent deformation such that the field strength superfields of $\mathcal{N} = 1/2$ super Yang-Mills theory are correctly reproduced by a curvature two-form superfield. We found the covariant constraints on the curvature two-form (referred to as Yang-Mills constraints) and the super ASD condition for the deformed ASD instantons.

Based on the deformed exterior algebra, we have formulated a non(anti)commutative version of the ADHM construction and shown that the resulting curvature two-form superfield indeed satisfies the Yang-Mills constraints as well as the super ASD condition. This means that our construction correctly gives deformed ASD instantons in the $\mathcal{N} = 1/2$ super Yang-Mills theory. We have seen that deformation terms emerge in the bosonic ADHM constraints (see also [24]), which are comparable with the U(1) terms due to space-space noncommutativity [28]. Our formulation reveals the geometrical meaning of those deformation terms as non(anti)commutativity of superspace.

The deformed super ADHM construction will facilitate us to discuss the moduli space of the deformed ASD instantons in the $\mathcal{N} = 1/2$ super Yang-Mills theory. In this paper, we saw that the numbers of the bosonic and fermionic moduli parameters of our solutions are the same as the ordinary theory: They are $4kn$ and $2kn$ respectively, where k is the instanton number. Additional moduli parameters, if they exist, may be contained in the $\theta\theta$-component of $\hat{\Delta}_\alpha$, but this would change, for example, the ADHM constraints, leading to a discrepancy with the result in [24]. We believe that our construction gives all the deformed ASD instantons, but it would be interesting to check it directly by considering reciprocity [29].

Finally we would like to give a comment on a relation between the deformed ADHM constraints and the hyper-Kähler quotient construction [30]. In the ordinary (commutative or noncommutative) gauge theory, the fermionic ADHM constraints ensure that the fermionic moduli are Grassmann-valued symplectic tangent vectors of the bosonic moduli space. Since the fermionic ADHM constraints are not modified in the present case, we expect that this interpretation is not modified. On the other hand, our bosonic ADHM constraints contain deformation terms which are $k \times k$ matrices, not just U(1) terms in general. If there is a U(1) term, it is well known that setting a particular value of the term corresponds to choosing a particular level set in the $n + 2k$ dimensional mother space in the hyper-Kähler quotient construction. The ordinary instantons or the localized instantons [31] correspond to this value set to be zero, and the Nekrasov-Schwarz instantons [28] correspond to this value set to be a non-zero constant. It needs a further study to clarify how our deformation terms can be interpreted in the hyper-Kähler quotient construction.

Acknowledgments

T.A. is very grateful to the members of the Particle Theory and Cosmology Group at Tohoku University for their support and hospitality. This research is partly supported by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No. 13640256 and 13135202. T.A. is supported by Grant-in-Aid for Scientific Research in Priority Areas (No. 14046201) from the Ministry of Education, Culture, Sports, Science and Technology.

A. Notation and conventions

We use the following sigma matrices:

$$\sigma^\mu = \sigma_\mu \equiv (-i\mathbb{1}, \sigma^i)\,, \qquad \bar\sigma^\mu = \bar\sigma_\mu \equiv (-i\mathbb{1}, -\sigma^i)\,, \tag{A.1}$$

where σ^i are the Pauli matrices and $\bar\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$ holds. The Lorentz generators are

$$\sigma^{\mu\nu} \equiv \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \qquad \bar\sigma_{\mu\nu} \equiv \frac{1}{4}(\bar\sigma_\mu\sigma_\nu - \bar\sigma_\nu\sigma_\mu)\,, \tag{A.2}$$

where

$$\sigma^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}\,, \qquad \bar\sigma^{\mu\nu} = -\frac{1}{2}\varepsilon^{\mu\nu\lambda\rho}\bar\sigma_{\lambda\rho}\,, \qquad \varepsilon^{0123} = \varepsilon_{0123} \equiv -1\,. \tag{A.3}$$

They can be written as

$$\sigma^{\mu\nu} = -\frac{i}{2}\eta^i_{\mu\nu}\sigma^i\,, \qquad \bar\sigma^{\mu\nu} = -\frac{i}{2}\bar\eta^i_{\mu\nu}\sigma^i \tag{A.4}$$

in terms of 't Hoot's eta symbol $\eta^i_{\mu\nu}$, $\bar\eta^i_{\mu\nu}$. The 't Hoot's eta symbol is defined by

$$\eta^i_{\mu\nu} \equiv (\epsilon_{i\mu\nu0} - \delta_{\mu0}\delta_{i\nu} + \delta_{\nu0}\delta_{i\mu})\,, \qquad \bar\eta^i_{\mu\nu} \equiv (\epsilon_{i\mu\nu0} + \delta_{\mu0}\delta_{i\nu} - \delta_{\nu0}\delta_{i\mu})\,. \tag{A.5}$$

We define

$$x_{\alpha\dot\beta} \equiv ix_\mu\sigma^\mu_{\alpha\dot\beta}\,, \qquad x^{\dot\beta\alpha} \equiv ix_\mu\bar\sigma^{\dot\beta\alpha}_\mu\,. \tag{A.6}$$

The "zero dimensional Dirac operator" in the extended ADHM construction is defined by

$$\hat\Delta_{\alpha[k]\times[n+2k]} = \Delta_\alpha(y) + \theta_\alpha\mathcal{M}\,, \tag{A.7}$$

where

$$\Delta_\alpha(x) \equiv a_\alpha + x_{\alpha\dot\beta}b^{\dot\beta} \tag{A.8}$$

and

$$
\begin{aligned}
a_{\alpha[k]\times[n+2k]} &\equiv \Big(\omega_{\alpha[k]\times[n]} (a'{}_{\alpha\dot{\beta}})_{[k]\times[2k]}\Big) = \Big((\omega_\alpha{}^i{}_u) \quad (a'{}_{\alpha\dot{1}}{}^i{}_j) \quad (a'{}_{\alpha\dot{2}}{}^i{}_j) \Big), \\
\mathcal{M}_{[k]\times[n+2k]} &\equiv \Big(\mu_{[k]\times[n]} \quad (\mathcal{M}'{}_{\dot{\beta}})_{[k]\times[2k]}\Big) = \Big((\mu^i{}_u) \quad (\mathcal{M}'{}_{\dot{1}}{}^i{}_j)(\mathcal{M}'{}_{\dot{2}}{}^i{}_j) \Big), \qquad \text{(A.9)}
\end{aligned}
$$

with $u = 1, \ldots, n$ and $i, j = 1, \ldots, k$. Note that we write $a'{}_{\alpha\dot{\beta}}{}^j{}_i = ia'^{\mu j}{}_i \sigma_{\mu\alpha\dot{\beta}}$. The canonical form of b is defined as

$$
(b^{\dot{\alpha}}) = \begin{pmatrix} b^{\dot{1}} \\ b^{\dot{2}} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{[k]\times[n]} & \mathbb{1}_k & \mathbf{0}_k \\ \mathbf{0}_{[k]\times[n]} & \mathbf{0}_k & \mathbb{1}_k \end{pmatrix}. \qquad \text{(A.10)}
$$

B. The inverse superfield

In this appendix, we give an expression of the "inverse" of a chiral superfield. Assume $(\phi_1{}^i{}_j)$ is an invertible $k \times k$ matrix. For a $k \times k$ matrix valued superfield

$$
\Phi(y, \theta) = \phi_1(y) + \sqrt{2}\theta\psi_1(y) + \theta\theta F_1(y)\,, \qquad \text{(B.1)}
$$

its "inverse" superfield $\Phi^{-1}(y, \theta)$ is defined by the relation

$$
\Phi * \Phi^{-1} = \Phi^{-1} * \Phi = \mathbb{1}_k\,. \qquad \text{(B.2)}
$$

Explicitly it is given by

$$
\Phi^{-1}(y, \theta) = \phi_2(y) + \sqrt{2}\theta\psi_2(y) + \theta\theta F_2(y)\,, \qquad \text{(B.3)}
$$

where

$$
\begin{aligned}
\phi_2 &= (\mathcal{A} + \det C \mathcal{F}\mathcal{A}^{-1}\mathcal{F})^{-1}\,, && \text{(B.4)} \\
\psi_{2\alpha} &= -\mathcal{A}'^{-1}(\Psi_\alpha + \varepsilon_{\alpha\beta}C^{\beta\gamma}\Psi_\gamma\mathcal{A}^{-1}\mathcal{F})\phi_2\,, && \text{(B.5)} \\
F_2 &= -\mathcal{A}^{-1}\mathcal{F}\phi_2\,. && \text{(B.6)}
\end{aligned}
$$

Here we have defined the following quantities:

$$
\begin{aligned}
\mathcal{A}' &\equiv \phi_1 + \det C F_1 \phi_1^{-1} F_1\,, && \text{(B.7)} \\
\Psi_\alpha &\equiv (\delta_\alpha^\beta - \varepsilon_{\alpha\gamma}C^{\gamma\beta}F_1\phi_1^{-1})\psi_{1\beta}\,, && \text{(B.8)} \\
\mathcal{A} &\equiv \phi_1 + C^{\alpha\beta}\psi_{1\alpha}\mathcal{A}'^{-1}\Psi_\beta\,, && \text{(B.9)} \\
\mathcal{F} &\equiv F_1 + \psi_1^\alpha \mathcal{A}'^{-1}\Psi_\alpha\,. && \text{(B.10)}
\end{aligned}
$$

C. Determination of the zero mode superfield

In this appendix, we give a detail of determination of the zero mode superfield $\hat{u}$ such that it correctly satisfies the normalization condition (4.33) and the WZ gauge conditions (4.67).

Let us begin with solving the zero mode eq. (4.32). This equation reads

$$\begin{aligned}0 = \Delta_\alpha u^{(0)} - \frac{1}{2}\mathcal{M}(\varepsilon C u^{(1)})_\alpha +& \\ + \theta^\beta \left(\Delta_\alpha u^{(1)}_\beta + \varepsilon_{\alpha\beta}\mathcal{M}u^{(0)} - C_{\alpha\beta}\mathcal{M}u^{(2)}\right) +& \\ + \theta\theta\left(\Delta_\alpha u^{(2)} + \frac{1}{2}\mathcal{M}u^{(1)}_\alpha\right),&\end{aligned} \tag{C.1}$$

where $(\varepsilon C)_\alpha{}^\beta \equiv \varepsilon_{\alpha\gamma}C^{\gamma\beta}$. With a given v that satisfies $\Delta_\alpha v = 0$ and $v^\ddagger v = \mathbb{1}_n$, $u^{(0)}$ and $u^{(2)}$ can be written with the use of $u^{(1)}_\alpha$ as

$$u^{(0)} = \frac{1}{2}(\Delta^\ddagger K)^\gamma \mathcal{M}(\varepsilon C u^{(1)})_\gamma + vc\,, \tag{C.2}$$

$$u^{(2)} = -\frac{1}{2}(\Delta^\ddagger K)^\gamma \mathcal{M}u^{(1)}_\gamma + vt\,, \tag{C.3}$$

where $(\Delta^\ddagger K)^\alpha \equiv \Delta^{\ddagger\gamma}K_\gamma{}^\alpha$ and c, t are arbitrary $n\times n$ bosonic matrices. Substituting (C.2) and (C.3) into (C.1), we find that the zero mode equation (C.1) becomes

$$\Delta_\alpha u^{(1)\beta} - \mathcal{M}u(\delta_\alpha{}^\beta c + (\varepsilon C)_\alpha{}^\beta t) - \frac{1}{2}\mathcal{M}(\Delta^\ddagger K)^{\gamma'}\mathcal{M}(\delta_\alpha{}^\beta(\varepsilon C)_{\gamma'}{}^\gamma - (\varepsilon C)_\alpha{}^\beta\delta_{\gamma'}{}^\gamma)u^{(1)}_\gamma = 0\,. \tag{C.4}$$

We will solve this equation. The $n+2k$ dimensional space is spanned by the $n+2k$ column vectors $\{\Delta^{\ddagger\alpha}, v\}$, thus in general, we can write $u^{(1)\alpha}$ as

$$u^{(1)\alpha} = v\gamma^\alpha + \Delta^{\ddagger\beta}r_\beta{}^\alpha\,. \tag{C.5}$$

where γ^α is a $k\times k$ fermionic matrix, and $r_\beta{}^\alpha$ is $k\times n$ one. As eq. (C.4) is composed of four independent equations with respect to the spinor indices, we can split it into one proportional to $\delta_\alpha{}^\beta$, one proportional to $(\varepsilon C)_\alpha{}^\beta$ and the other two equations. Substituting the general form (C.5) into (C.4), we find that the latter two equations are

$$f^{-1}r'_\alpha{}^\beta + \frac{1}{2}(\varepsilon C)_\alpha{}^\gamma r'_\gamma{}^\beta \mathcal{M}\mathcal{M}^\ddagger = 0\,, \tag{C.6}$$

where $r'_\alpha{}^\beta$ represents the terms contained in $r_\alpha{}^\beta$ which are not proportional to $\delta_\alpha{}^\beta$ or $(\varepsilon C)_\alpha{}^\beta$. Because of the explicit C-dependence of the second term in the l.h.s., we find that $r'_\alpha{}^\beta$ should vanish by a perturbative argument with respect to C.[3] Thus the form of $u^{(1)\alpha}$ is simplified as

$$u^{(1)\alpha} = v\gamma^\alpha + \Delta^{\ddagger\alpha}r + \Delta^{\ddagger\beta}(\varepsilon C)_\beta{}^\alpha s\,, \tag{C.7}$$

where r and s are $k\times n$ fermionic matrices. Substituting this equation into eq. (C.4), r and s are written in terms of c, t and γ^α. As a result, the zero mode equation (C.1) is solved by (C.2)–(C.4) with

$$\begin{aligned}\begin{pmatrix} r \\ s\end{pmatrix} &= Z^{-1}f(1+\frac{1}{4}\det C(\mathcal{M}vv^\ddagger\mathcal{M}^\ddagger Z^{-1}f)^2)^{-1}\times \\ &\times\begin{pmatrix} 1 & \frac{1}{2}\det C\mathcal{M}vv^\ddagger\mathcal{M}^\ddagger Z^{-1}f \\ -\frac{1}{2}\mathcal{M}vv^\ddagger\mathcal{M}^\ddagger Z^{-1}f & 1\end{pmatrix}\times \\ &\times\begin{pmatrix}\mathcal{M}vc + \frac{1}{2}\mathcal{M}\Delta^{\ddagger\alpha}K_\alpha{}^\beta(\varepsilon C)_\beta{}^\gamma\mathcal{M}v\gamma_\gamma \\ \mathcal{M}vt - \frac{1}{2}\mathcal{M}\Delta^{\ddagger\alpha}K_\alpha{}^\beta\mathcal{M}v\gamma_\beta.\end{pmatrix}.\end{aligned} \tag{C.8}$$

[3] We assume that the deformation parameter dependence admits a perturbative expansion.

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Next, we consider the WZ gauge fixing condition (4.67), $\hat{u}^{\ddagger} * \partial_\alpha \hat{u} = 0$. This equation reads

$$
\begin{aligned}
u^{(0)\ddagger} u^{(1)\alpha} &= C^{\alpha\beta} u^{(1)\ddagger}_{\beta} u^{(2)} \,, && \text{(C.9)} \\
u^{(0)\ddagger} u^{(2)} &= \frac{1}{4} u^{(1)\ddagger\gamma} u^{(1)}_{\gamma} \,, && \text{(C.10)} \\
C_{\alpha\beta} u^{(2)\ddagger} u^{(2)} &= -\frac{1}{2} u^{(1)\ddagger}_{(\alpha} u^{(1)}_{\beta)} \,, && \text{(C.11)} \\
u^{(1)\ddagger}_{\alpha} u^{(2)} &= u^{(2)\ddagger} u^{(1)}_{\alpha} \,. && \text{(C.12)}
\end{aligned}
$$

From eqs. (C.3), (C.5), (C.9) and (C.12) we find

$$
\gamma^\alpha = \frac{1}{2} (\mathbb{1}_2 \otimes c^{\ddagger} + 2\mathcal{C} \otimes t^{\ddagger})^{-1}{}_{\beta}{}^{\alpha} u^{(1)\ddagger\gamma'} \mathcal{M}^{\ddagger} \left((\varepsilon C)_{\gamma'}{}^{\sigma} r_{\sigma}{}^{\beta} - r_{\gamma'}{}^{\sigma} (\varepsilon C)_{\sigma}{}^{\beta} \right) . \tag{C.13}
$$

This equation tells us that γ^α vanishes if $r_\alpha{}^\beta$ has the form $\delta_\alpha{}^\beta r + (\varepsilon C)_\beta{}^\alpha s$ as in (C.7). Eq. (C.7) with $\gamma^\alpha = 0$ solves the equations (C.9) and (C.12).

Then the remaining zero mode equation for $u^{(1)}_\alpha$ gives the following two independent equations which correspond to (C.10) and (C.11):

$$
\begin{aligned}
c^{\ddagger} t + \begin{pmatrix} r^{\ddagger} & s^{\ddagger} \end{pmatrix} \begin{pmatrix} \frac{1}{2} f^{-1} Z & -\frac{1}{4} \det C \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} \\ -\frac{1}{4} \det C \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} & -\frac{1}{2} \det C f^{-1} Z \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} &= 0 \,, && \text{(C.14)} \\
t^{\ddagger} t - \begin{pmatrix} r^{\ddagger} & s^{\ddagger} \end{pmatrix} \begin{pmatrix} -\frac{1}{4} \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} & -\frac{1}{2} f^{-1} Z \\ -\frac{1}{2} f^{-1} Z & \frac{1}{4} \det C \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} &= 0 \,, && \text{(C.15)}
\end{aligned}
$$

where r and s are written in terms of c and t as (C.8) with $\gamma^\alpha = 0$. We first consider eq. (C.14). Substituting eq. (C.8) into (C.14), we obtain

$$
\begin{aligned}
c^{\ddagger} t - \frac{1}{2} \begin{pmatrix} c^{\ddagger} & t^{\ddagger} \end{pmatrix} v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \begin{pmatrix} 1 & \frac{1}{2} \det C \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \\ \frac{1}{2} \det C \mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f & -\det C \end{pmatrix} \times \quad \text{(C.16)} \\
\times \left(1 + \frac{1}{4} \det C \left(\mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \right)^2 \right)^{-1} \mathcal{M} v \begin{pmatrix} c \\ t \end{pmatrix} = 0 \,.
\end{aligned}
$$

To simplify this equation, it is useful to rewrite t as

$$
t = \frac{1}{2} v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \mathcal{M} v c + t' \,. \tag{C.17}
$$

Using eq. (C.17), we find that eq. (C.16) becomes

$$
\left(c^{\ddagger} + \frac{1}{2} \det C t'^{\ddagger} v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \left(1 + \frac{1}{4} \det C \left(\mathcal{M} v v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \right)^2 \right)^{-1} \mathcal{M} v \right) t' = 0. \tag{C.18}
$$

Note that c cannot be proportional to $\det C$ when we consider perturbative solutions, because $c = 1$ in the undeformed case. Thus the bracket in front of t' cannot be zero, and t' must vanish. As a result, we obtain

$$
t = \frac{1}{2} v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \mathcal{M} v c. \tag{C.19}
$$

We can also verify that this satisfies eq. (C.15). Now r and s can be obtained by substituting (C.19) into (C.8):

$$r = Z^{-1} f \mathcal{M} v c\,, \qquad s = 0\,. \tag{C.20}$$

So far we have solved the zero mode equation (C.1) and imposed the gauge fixing conditions (C.9)–(C.12). The remaining freedom is c. This is fixed by the normalization condition for the zero mode $\hat{u}$, as we will see below. The normalization condition is eq.~(4.33):

$$\begin{aligned}
\mathbb{1}_n &= u^{(0)\ddagger} u^{(0)} - \frac{1}{2} C^{\alpha\beta} u^{(1)\ddagger}_{\alpha} u^{(1)}_{\beta} - \det C u^{(2)\ddagger} u^{(2)} + \\
&\quad + \theta^{\alpha} \left(u^{(1)\ddagger}_{\alpha} u^{(0)} + u^{(0)\ddagger} u^{(1)}_{\alpha} - \varepsilon_{\alpha\beta} C^{\beta\gamma} (u^{(1)\ddagger}_{\gamma} u^{(2)} - u^{(2)\ddagger} u^{(1)}_{\gamma}) \right) + \\
&\quad + \theta\theta \left(u^{(2)\ddagger} u^{(0)} + u^{(0)\ddagger} u^{(2)} - \frac{1}{2} u^{(1)\ddagger\gamma} u^{(1)}_{\gamma} \right).
\end{aligned} \tag{C.21}$$

The only non-trivial condition comes from the lowest component, because the higher components in the r.h.s. vanish automatically if (C.9) and (C.10) are satisfied. It leads us to

$$c = N^{-\frac{1}{2}} = (1 + \frac{1}{4} \det C (v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \mathcal{M} v)^2)^{-\frac{1}{2}}\,. \tag{C.22}$$

After some algebra, we obtain the zero mode $\hat{u}$ which satisfies simultaneously the WZ gauge conditions and the normalization condition:

$$\begin{aligned}
\hat{u} &= (1 + \Delta^{\ddagger\alpha} f \mathcal{M} \theta + \frac{1}{2} \mathcal{M}^{\ddagger} f \mathcal{M} \theta\theta) u^{(0)} && \text{(C.23)} \\
u^{(0)} &= (1 - \frac{1}{2} \Delta^{\ddagger\alpha} K_{\alpha}{}^{\beta} (\varepsilon C)_{\beta}{}^{\gamma} \Delta_{\gamma} \mathcal{M}^{\ddagger} Z^{-1} f \mathcal{M}) v N^{-\frac{1}{2}} U && \text{(C.24)} \\
N &= 1 + \frac{1}{4} \det C (v^{\ddagger} \mathcal{M}^{\ddagger} Z^{-1} f \mathcal{M} v)^2\,, && \text{(C.25)}
\end{aligned}$$

where U is a unitary matrix such that $U^{\dagger} = U^{-1}$, and K, Z, N are given by (4.71), (4.74), (4.75), respectively.

D. U(2) one instanton

In this appendix, we construct deformed super instantons in the case of U(2) gauge group and instanton number $k = 1$ with the use of the deformed super ADHM construction in section 4. We will find our solutions are consistent with the results in refs. [21 – 24].

Let us begin with solving the deformed ADHM constraints (4.45) and (4.46) and express the constrained ADHM data a_{α} and $\mathcal{M}$ in terms of unconstrained free parameters. In the U(2) $k = 1$ case, the deformed ADHM constraints (4.45) become

$$\sigma^{i}_{\beta}{}^{\alpha} \omega_{\alpha\dot{\alpha}} \omega^{\ddagger\dot{\alpha}\beta} + \frac{i}{2} C^{i} \mathcal{M} \mathcal{M}^{\ddagger} = 0\,, \tag{D.1}$$

$$\omega_{\alpha\dot{\alpha}} \mu^{\ddagger\dot{\alpha}} + \varepsilon_{\alpha\beta} \mu_{\dot{\alpha}} \omega^{\ddagger\dot{\alpha}\beta} = 0\,. \tag{D.2}$$

Here we have defined $C^{i} \equiv \frac{1}{2} \eta^{i}_{\mu\nu} C^{\mu\nu}$ ($(C^{i})^{\ddagger} = -C^{i}$) where $\eta^{i}_{\mu\nu}$ is the 't Hooft eta symbol (see appendix A). Note that in this section we denote the U(2) gauge index u in (A.9) as a dotted spinor index ($\dot{\alpha}$, $\dot{\beta}$, etc.).

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These constraints are solved by

$$\mu^{\ddagger\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\rho^{-1}\zeta^{\alpha}\omega_{\alpha\dot{\beta}}\,, \qquad \mathcal{M}'^{\ddagger\dot{\alpha}} = \bar{\xi}^{\dot{\alpha}}\,, \qquad \omega_{\alpha\dot{\alpha}} = \omega_{\mu} i\sigma^{\mu}_{\alpha\dot{\alpha}}\,, \tag{D.3}$$

where

$$\omega_{\mu} \equiv \left(\rho, -\frac{1}{8}\rho^{-1}C^{i}(\bar{\xi}\bar{\xi} - \zeta\zeta)\right), \tag{D.4}$$

$\rho^{\ddagger} = \rho$, $(\zeta_{\alpha})^{\ddagger} = \zeta^{\alpha}$, $(\bar{\xi}_{\dot{\alpha}})^{\ddagger} = \bar{\xi}^{\dot{\alpha}}$, and the remaining parameter a'_{μ} can be arbitrary. The parameters ρ, $\bar{\xi}_{\dot{\alpha}}$ and ζ_{α} are unconstrained free parameters and correspond to the scale, supersymmetry and superconformal moduli, respectively.[4]

Up to now, we have solved the deformed ADHM constraints and found that the operator $\hat{\Delta}_{\alpha}$ is written in terms of the unconstrained moduli parameters ρ, $\bar{\xi}$ and ζ. In order to construct the connection ϕ of the deformed instanton, our next task is to determine the normalized zero mode $\hat{u}$ of $\hat{\Delta}_{\alpha}$, or $u^{(0)}$ appearing in eq. (4.78).

As we have mentioned in the previous section, the operator Δ_{α} (the fermionic moduli $\mathcal{M}$) can be split into the C-independent part $\Delta_{0\alpha}$ ($\mathcal{M}_0$) and the residual C-dependent part $\delta\Delta_{\alpha}$ ($\delta\mathcal{M}$) (see eq. (4.76)), and the normalized zero mode v of Δ_{α} is given by these quantities as in eq. (4.77). Because the deformed ADHM constraints are solved by (D.3), they are now explicitly written with the use of

$$\rho_{\mu} \equiv (\rho, \mathbf{0})\,, \qquad \delta\rho_{\mu} \equiv \left(0, -\frac{1}{8}\rho^{-1}C^{i}(\bar{\xi}\bar{\xi} - \zeta\zeta)\right) \tag{D.5}$$

as

$$\Delta_{0\alpha} = (\,\rho_{\alpha\dot{\alpha}} \quad y_{\alpha\dot{\beta}}\,)\,, \qquad \delta\Delta_{\alpha} = (\,\delta\rho_{\alpha\dot{\alpha}} \quad \mathbf{0}\,)\,, \tag{D.6}$$

$$\mathcal{M}_0 = (\,\rho^{-1}\zeta^{\alpha}\rho_{\alpha\dot{\alpha}} \quad \bar{\xi}_{\dot{\alpha}}\,)\,, \qquad \delta\mathcal{M} = -\rho^{-1}\zeta^{\alpha}\delta\Delta_{\alpha}\,, \tag{D.7}$$

where $\rho_{\alpha\dot{\alpha}} \equiv \rho_{\mu} i\sigma^{\mu}_{\alpha\dot{\alpha}}$, $\delta\rho_{\alpha\dot{\alpha}} \equiv \delta\rho_{\mu} i\sigma^{\mu}_{\alpha\dot{\alpha}}$. Note that here we have absorbed the translation moduli $a'_{\mu[1]\times[1]}$ into y_{μ}. Then the zero mode $\hat{u}$ can be written in terms of $\Delta_{0\alpha}$, $\delta\Delta_{\alpha}$, $\mathcal{M}_0$, $\delta\mathcal{M}$ and the normalized zero mode v_0 of $\Delta_{0\alpha}$. Here we choose the U(2) $k=1$ instanton in the non-singular gauge (the BPST instanton) as v_0:[5]

$$v_0 = \rho^{-1}(y^2+\rho^2)^{-1/2}\begin{pmatrix} \rho^{\dot{\alpha}\alpha}y_{\alpha\dot{\beta}} \\ -\rho^2\delta^{\dot{\gamma}}_{\ \dot{\beta}} \end{pmatrix}. \tag{D.8}$$

Then from eq. (4.77), we obtain v as

$$\begin{aligned} v = v_0 &- \rho^{-1}(y^2+\rho^2)^{-3/2}\begin{pmatrix} -\rho^2\delta\rho^{\dot{\alpha}\alpha}y_{\alpha\dot{\beta}} \\ y^{\dot{\gamma}\gamma}\delta\rho_{\gamma\dot{\gamma}'}\rho^{\dot{\gamma}'\gamma'}y_{\gamma'\dot{\beta}} \end{pmatrix} - \\ &- \frac{1}{16}\det C\zeta\zeta\bar{\xi}\bar{\xi}\rho^{-3}(y^2+\rho^2)^{-5/2}\begin{pmatrix} (y^2-2\rho^2)\rho^{\dot{\alpha}\alpha}y_{\alpha\dot{\beta}} \\ -3\rho^2y^2\delta^{\dot{\gamma}}_{\ \dot{\beta}} \end{pmatrix}. \end{aligned} \tag{D.9}$$

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[4]Our superconformal moduli parameter ζ is different from the one in refs. [22, 23] which corresponds to $\rho^{-1}\zeta$ in our convention.

[5]Instead of this, if we choose the 't Hooft instanton as v_0,

$$v_0 = \begin{pmatrix} |y|(y^2+\rho^2)^{-\frac{1}{2}}\delta^{\dot{\alpha}}_{\ \dot{\beta}} \\ -|y|^{-1}(y^2+\rho^2)^{-\frac{1}{2}}y^{\dot{\alpha}\alpha}\rho_{\alpha\dot{\beta}} \end{pmatrix},$$

then we can construct the singular deformed super instantons in ref. [24].

Next we express $u^{(0)}$ defined in eq. (4.73) in terms of v_0, $\Delta_{0\alpha}$, $\delta\Delta_\alpha$, $\mathcal{M}_0$ and $\delta\mathcal{M}$. Note that in the U(2) $k=1$ case, the following equation holds:

$$\left(v_0{}^\ddagger \mathcal{M}_0{}^\ddagger \mathcal{M}_0 v_0\right)^2 = 0\,. \tag{D.10}$$

This equation will turn out to make the almost all quantities vanish in our calculation. To verify this equation, we need to use v_0 in (D.8). Together with the form of $\mathcal{M}_0$ in (D.7), the above equation (D.10) is easily shown. First, the inverse of Z in (4.74) is found to be

$$Z^{-1} = 1 - \frac{1}{4}\det C f_0 \mathcal{M}_0 \Delta_0^{\ddagger\alpha} f_0 \mathcal{M}_0 \mathcal{M}_0^\ddagger f_0 \Delta_{0\alpha} \mathcal{M}_0^\ddagger + \cdots \tag{D.11}$$

by using eq. (4.71), where $\cdots$ denotes the terms depending on more than four fermionic moduli. Note that in the case of U(2) $k=1$, there are only four fermionic moduli parameters, so that the $\mathcal{O}(\mathcal{M}_0{}^5)$ terms vanish automatically. Here we have used the equation $\mathcal{M}\Delta^{\ddagger\alpha} f(\varepsilon C)_\alpha{}^\beta \Delta_\beta \mathcal{M}^\ddagger = 0$, which follows from the fermionic ADHM constraint, the symmetric property of $C^{\alpha\beta}$ and the fact that $\mathcal{M}\Delta^{\ddagger\alpha}$ and $\Delta_\alpha\mathcal{M}^\ddagger$ anticommute in the $k=1$ case. Then, due to eq. (D.10), the normalization factor N (4.75) becomes $N = \mathbb{1}_2 + \mathcal{O}(\mathcal{M}^8)$. As a result, we find an expression of $u^{(0)}$ from eq. (4.73) as

$$u^{(0)} = \left(\mathbb{1}_4 - \frac{1}{2}\Delta^{\ddagger\alpha} f(\varepsilon C)_\alpha{}^\beta \Delta_\beta \mathcal{M}^\ddagger f \mathcal{M} - \frac{1}{4}\det C \Delta_0^{\ddagger\alpha} f_0 \mathcal{M}_0 \mathcal{M}_0^\ddagger f_0 \Delta_{0\alpha} \mathcal{M}_0^\ddagger f_0 \mathcal{M}_0\right) v\,. \tag{D.12}$$

Here we have taken $U = \mathbb{1}_2$ for simplicity.

Substituting eq. (D.12) with eq. (4.77) into the eq. (4.78), we obtain the connection superfield after a lengthy but straightforward calculation:

$$\begin{aligned}(\phi_\mu)^{\dot\beta}{}_{\dot\gamma} =& -2(y^2+\rho^2)^{-1}\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\gamma} y_\nu + \\ &+ \frac{1}{4}C_{\mu\nu}\partial_\nu\left(K_1\bar\xi\bar\xi + K_2\zeta\zeta + 2\rho K_3\right) - \frac{1}{4}\det C\rho^{-2}(y^2+\rho^2)^{-2}\zeta\zeta\bar\xi\bar\xi\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\gamma} y_\nu + \\ &+ \rho^2(y^2+\rho^2)^{-2}(\rho^{-1}\zeta^\gamma y_{\gamma\dot\gamma'} + \bar\xi_{\dot\gamma'})(\varepsilon^{\dot\beta\dot\gamma'}\varepsilon_{\dot\gamma\dot\alpha} + \delta^{\dot\beta}{}_{\dot\alpha}\delta^{\dot\gamma'}{}_{\dot\gamma})\bar\sigma_\mu^{\dot\alpha\alpha}\theta_\alpha\,,\end{aligned} \tag{D.13}$$

where

$$K_1 = \frac{y^2}{(y^2+\rho^2)^2} - \frac{2}{y^2+\rho^2}\,, \qquad K_2 = \frac{y^2}{(y^2+\rho^2)^2} + \frac{1}{y^2+\rho^2}\,, \qquad K_3 = \frac{\zeta\sigma_\sigma\bar\xi y^\sigma}{(y^2+\rho^2)^2}\,. \tag{D.14}$$

From this connection superfield, we can read the fermion zero mode and the gauge field of the deformed super instanton:

$$(\bar\lambda_{\dot\alpha})^{\dot\beta}{}_{\dot\gamma} = -2i\rho^2(y^2+\rho^2)^{-2}(\rho^{-1}\zeta^\gamma y_{\gamma\dot\gamma'} + \bar\xi_{\dot\gamma'})(\varepsilon^{\dot\beta\dot\gamma'}\varepsilon_{\dot\gamma\dot\alpha} + \delta^{\dot\beta}{}_{\dot\alpha}\delta^{\dot\gamma'}{}_{\dot\gamma})\,, \tag{D.15}$$

$$\begin{aligned}(v_\mu)^{\dot\beta}{}_{\dot\gamma} =& -4i(y^2+\rho^2)^{-1}\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\gamma} y_\nu + \frac{i}{2}C_{\mu\nu}\partial_\nu\left(K_1\bar\xi\bar\xi + K_2\zeta\zeta + 2\rho K_3\right)\delta^{\dot\beta}{}_{\dot\gamma} - \\ &- \frac{i}{2}\det C\rho^{-2}(y^2+\rho^2)^{-2}\zeta\zeta\bar\xi\bar\xi\bar\sigma_{\mu\nu}{}^{\dot\beta}{}_{\dot\gamma} y_\nu\,.\end{aligned} \tag{D.16}$$

Note that the fermion zero mode coincides with the one in the undeformed case, which is consistent with the result in refs. [22, 23]. On the other hand, the above gauge field does not coincide with the one in refs. [22, 23] by the term proportional to $\det C$. This does not mean that our result contradicts the known results. As we have shown in the previous sections, our deformed ADHM construction correctly gives the solutions to the deformed ASD equations. We should note, however, that there is a freedom how we parameterize the instanton moduli space. In fact, by re-parameterizing the scale parameter in our solution as

$$\rho \to \rho \left(1 - \frac{1}{16} \det C \rho^{-4} \zeta\zeta\bar{\xi}\bar{\xi}\right), \tag{D.17}$$

we find that our solution becomes consistent with the one obtained in refs. [22, 23].

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